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LETTER TO THE EDITOR

Quenching of the Aharonov–Bohm oscillations in variable cross section geometries

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Abstract. The electronic spectrum of an infinite quasi-one-dimensional disorder-free system, of varying width, is considered. It is shown that as a result of the varying width, the spectrum includes extended states as well as states which are exponentially decaying. The effect of magnetic fields is considered and criteria for the weak and strong field regimes are derived. It is shown that the amplitude of the Aharonov–Bohm oscillations in multiply-connected systems can be exponentially small due to the spatial confinement of the probability density. Physical implications for mesoscopic samples are discussed.

There is currently considerable interest in the electronic properties of low-dimensional and mesoscopic structures. Interference effects have been studied in great detail (Imry 1986, Washburn and Webb 1986) and particular attention has been paid to magnetic-flux effects in multiply connected conductors (Aronov and Sharvin 1987). The theoretical considerations are usually made for ideally shaped geometries, e.g. strictly one-dimensional rings or cylinders, or rectangular long wires of constant width. It is rather obvious, however, that experimental samples are seldom of such perfect shapes. One expects that a real wire will possess a small degree of width variation or curvature, and that a fabricated ring will not be a perfect circle. The random roughness of a sample surface is usually included in the general treatment of disorder effects. This is not the case for a smooth regularly-shaped surface. These types of surfaces and boundaries have a profound effect upon the electronic spectrum of a low-dimensional sample. It is thus conceivable that experiments designed to test theoretical predictions based upon ideally shaped structures will fail to exhibit those predictions, not for conventional reasons such as finite-temperature and disorder effects (including surface scattering), but because the fabricated samples are of 'non-perfect', though regular, geometries.

In this letter we address the question of the electronic wavefunctions and energy spectrum of such structures. We include the effect of a constant magnetic field, and derive criteria for the field strength. In particular, we discuss the Aharonov–Bohm effect (Aharonov and Bohm 1959) of multiply connected geometries in the weak-field regime.

In a multiply connected layer, the wavefunction is obviously periodic with the longitudinal coordinate, the period being the perimeter L. Thus, quantum states are

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the Bloch ones. We demonstrate that when L is large compared with the (varying) shell width, the Bloch effective mass and bandwidth decrease exponentially with L. When $L \to \infty$ (and the system of an infinite period is a non-periodic one), the electronic spectrum of a system, *which is free of disorder*, consists of exponentially decaying 'localised' states and extended states, with a multitude of 'mobility edges' separating them.

The spectrum is characterised by two quantum numbers, one referring to the coordinate along the sample and the other related to the transverse coordinate. The latter quantum number, n, divides the spectrum into 'bands'. States in the lowest band ($n=1$) are localised up to the mobility edge of the first band, above which they are extended. States in the second band are localised below the mobility edge of the second band, and extended above it, and so on for higher n values. The widths of the localised state bands depend upon the ratio of the system cross section size to its length and generally *increase* as this ratio tends to zero. Thus, below a certain Fermi energy (i.e. below a certain critical electron concentration) all states will be localised. An increase in E_F leads to an insulator-metal transition at the first mobility edge E_1, and thence to metallic conductivity. Further mobility edges yield jumps (c.f. van Wees *et al* 1988, Wharam *et al* 1988, Glazman *et al* 1988) in the density of extended states. Of the many implications inferred by this conclusion, we dwell in the following upon the Aharonov-Bohm oscillations in narrow rings and show that their amplitude tends exponentially to zero below the first mobility edge.

A possible starting point would have been to consider the Schrödinger equation of a long wire (along the x direction) of a small cross section which is some function of x. However, a two-dimensional elliptic shell is more attractive, as it involves an orthogonal set of coordinates, allowing for explicit analytical solutions. That is why, for the sake of simplicity, we confine ourselves to the discussion of an elliptic shell threaded by a magnetic field H.

The Schrödinger equation in elliptic coordinates ($x = d(\cosh\theta\cos\phi)$, $y = d(\sinh\theta\sin\phi)$) is

$$\left\{\left[\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{2}\left(\frac{d}{L_0}\right)^2 \sin 2\phi\right]^2 + \left[\frac{\partial}{\partial\phi} - \frac{\mathrm{i}}{2}\left(\frac{d}{L_0}\right)^2 \sinh 2\theta\right]^2 + (kd)^2(\sinh^2\theta + \sin^2\phi)\right\}\psi(\theta,\phi) = 0 \tag{1}$$

where the energy is related to k^2, $E = \hbar^2k^2/2m$, and L_0 is the magnetic length, $L_0^2 = 2\hbar c/eH$. Substituting for the wavefunction the form

$$\psi = \exp\left\{\left[\frac{\mathrm{i}}{2}\left(\frac{d}{L_0}\right)^2 \sinh 2\theta_0\right]\phi\right\}\exp\left\{\left[\frac{\mathrm{i}}{2}\left(\frac{d}{L_0}\right)^2 \sin 2\phi\right](\theta-\theta_0)\right\}\Phi \tag{2}$$

and introducing the transverse coordinate z along the shell width

$$\theta = \theta_0 + \eta z \qquad 0 \leqslant z \leqslant 1 \tag{3}$$

we obtain the Schrödinger equation in the simple form

$$\left\{\left(\frac{\partial}{\eta\partial z}\right)^2 + \left[\frac{\partial}{\partial\phi} - \mathrm{i}2\eta z\left(\frac{d}{L_0}\right)^2 f(\phi)\right]^2 + (kd)^2 f(\phi)\right\}\Phi(z,\phi) = 0 \tag{4}$$

where we have denoted $f(\phi) = \cosh^2\theta_0 - \cos^2\phi$. In deriving (4) we have kept terms to order z and neglected $\eta z(d^2/2)\sinh(2\theta_0)$ compared to $d^2 f(\phi)$. The parameter η measures the ratio of the sample width to its length: denoting the inner axes of the shell by $a = d(\cosh\theta_0)$ and $b = d(\sinh\theta_0)$, respectively, and the outer axes by $a+\Delta a$ and $b+\Delta b$, respectively, we find $\eta \sim \Delta b/a \sim \Delta a/b$, i.e. η is very small for a thin shell or when the perimeter is very long. Thus the system becomes one-dimensional in the $\eta \to 0$ limit.

Consider first the solutions in the absence of the field. The wavefunction becomes $\Phi(z, \phi) = R(z)F(\phi)$. Applying the boundary conditions $R(0) = R(1) = 0$, we obtain $R(z) = \sin(\pi n z)$ and the following equation for $F(\phi)$:

$$\begin{aligned} &[\eta^2 d^2/d\phi^2 + q(\phi)]F(\phi) = 0 \\ &q(\phi) = (\eta k d)^2 f(\phi) - (\pi n)^2 \qquad n = 1, 2, \ldots . \end{aligned} \tag{5}$$

This is the Mathieu equation (Morse and Feshbach 1953). Equation (5) is of the same form as the Schrödinger equation, with η playing the role of the Planck constant. We consider a Bloch situation: $F(\phi+\pi) = \exp(\mathrm{i}\pi l)F(\phi)$, where l is an integer. Below the potential barrier top $(\pi n/\eta a < k < \pi n/\eta b)$, because $-q_{\min}/\eta^2 \gg 1$, the Bloch spectrum corresponds to the tight-binding picture. Then F decays exponentially in the classically forbidden regions $-\phi_1 < \phi < \phi_1$, $\phi_2 < \phi < \pi + \phi_1$, where $q(\phi_1) = q(\phi_2) = 0$. For example, for $0 < \phi < \phi_1$,

$$F(\phi) \propto \exp\left(-\eta^{-1}\int_0^{\phi} |q(\phi')|^{1/2}\,\mathrm{d}\phi'\right). \tag{6}$$

The spectrum is determined by

$$\begin{aligned} &\cos\left(\eta^{-1}\int_{\phi_1}^{\phi_2} \mathrm{d}\phi |q(\phi)|^{1/2}\right) = \cos(l\pi)/\cosh(\eta^{-1}\alpha) \\ &\alpha = \int_0^{\phi_1} d\phi(|q(\phi)|^{1/2}) + \eta \ln 2 \end{aligned} \tag{7}$$

i.e., the Bloch bandwidth and the Bloch effective mass are exponentially small as $\eta \to 0$.

At $k > \pi n/\eta b$ the classical orbits extend over all ϕ ('classical' mobility edge). When $|k - \pi n/\eta b| \sim \eta$, the spectrum is very complicated, but this region vanishes as $\eta \to 0$. When $|k - \pi n/\eta b| \gg \eta$,

$$F \propto \exp\left(\mathrm{i}\eta^{-1}\int_{\phi_1}^{\phi} [q(\phi')]^{1/2}\,\mathrm{d}\phi'\right) \tag{8}$$

and the spectrum is given by

$$\int_0^{\pi} |q(\phi)|^{1/2}\,\mathrm{d}\phi = \eta\pi l \tag{9}$$

leading to $k^2 = 2[(\pi n/\eta)^2 + l^2]/(a^2+b^2)$ in the $b \gg \pi n/\eta k$ limit.

Since $\eta \propto L^{-1}$ (L is the ellipse perimeter), the picture that emerges from equations (6)-(9) is similar to the Anderson localisation. All Bloch states are of course extended, but at $k < \pi n/\eta b$ the probability density decreases exponentially with the distance from the 'localisation regions' $\phi_1 < \phi < \phi_2$, $-\phi_2 < \phi < -\phi_1$. When $L \to \infty$, the bandwidth becomes proportional to $\exp\{-L/\xi\} \to 0$, ξ being provided by (6). Hence, starting at low energies, the states characterised by n are localised up to the mobility edge at $k^2 \sim (n\pi/\eta b)^2$, while the states belonging to $n-1$, $n-2, \ldots$ are extended. Above this energy the states corresponding to n also become extended, up to an energy of the order $k^2 \sim [(n+1)\pi/\eta a]^2$ above which states belonging to $n+1$ are localised. The width of the localised state bands is of the order $(nd/\eta ab)^2$, increasing as η tends to zero, while the gaps in between are of the order $[(n+1)/\eta a]^2-(n/\eta b)^2$ (as long as this quantity is positive).

Next we turn to the effect of a magnetic field. One notes from (4) that in the small-η limit the magnetic field is considered to be weak when $\eta(d/L_0)^2$ is smaller than unity and becomes stronger as this quantity increases. This means that as long as the magnetic length is much longer than $d\sqrt{\eta}$, the magnetic field term in (4) is small and can be handled by perturbation theory. In this regime the main effect of the magnetic field is brought about by the first phase factor in (2). This phase factor can be rewritten in the form $\exp(\mathrm{i}\phi N)$, where N is the magnetic flux inside the elliptic shell, divided by the flux quantum unit ϕ_0, $N=(d^2/2L_0^2)\sinh 2\theta_0=HS/\phi_0$. Here S is the area contained inside the *inner* curvature of the shell, $S=\pi ab$. For higher fields such that $(\eta d/L_0)^2$ is finite, i.e. the magnetic length is of the order of the shell width, perturbative treatment of (4) is invalid. In this regime there appear magnetic 'edge' states (Dingle 1953, Prange and Nee 1968). The investigation of the electronic spectrum and magnetic properties pertaining to this case will be presented elsewhere. In the following we concentrate upon the weak-field regime.

When the field is weak enough so that $\eta(d/L_0)^2<1$, then to lowest order the wavefunction solving (4) can still be written as $\Phi(z,\phi)=R(z)F(\phi)$, with $F(\phi)$ satisfying (5). However, the periodicity condition imposed on the wavefunction ψ is modified by the phase factor $\exp(\mathrm{i}\phi N)$ (see (2)). Consequently, the eigenvalue equations are modified as well, with l in (7) and (9) replaced by $(N-l)$. The magnetic-field effect upon the extended states (9) is the same as that found in strictly one-dimensional treatments of multiply connected systems (Gunther and Imry 1969, Buttiker *et al* 1983, 1984, Landauer and Buttiker 1985, Cheung *et al* 1988), leading to the Aharonov–Bohm oscillations (of unit amplitude) and persistent normal currents. However, for the low-lying energy states which are localised and correspond to an insulator, the Aharonov–Bohm effect decreases exponentially ((6) and (7)).

Our approach is readily generalised to other geometries, e.g. a three-dimensional wire, or arbitrarily varying (regularly or randomly) cross sections. In all cases the transverse momentum quantisation (of order w^{-1}, where w is the characteristic width) leads at the bottom of each band to the 'ballistic' localisation length $\xi_w\sim w\sqrt{w/\delta w}$, δw being a measure for the width variance. Any fixed width (or thickness) variation δw (at zero temperature) will lead to the probability density ρ of the ground state (and adjacent states) which decays exponentially as the length L tends to infinity, $\ln\rho\sim -L/\xi_w$.

In summary, an infinite quasi-one-dimensional system (even without disorder, but with regularly changing cross section) which has a sufficiently low bulk Fermi energy, E_F, will exhibit characteristics of strong localisation (like the metal–insulator transition), jumps in the density of extended states, etc. The peculiar features of the spectrum may be most conveniently probed by observing the effect of a magnetic field, which can be also used to tune the Fermi energy.

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